Alternating Series :-

The Integral test and the Comparison test given in previous lectures, Apply only to series with positive terms.

A series of the form $\sum_{n=1}^{\infty}(-1)^n x_n$ or $\sum_{n=1}^{\infty}(-1)^{n+1} x_n$, where $x_n > 0$ for all n, is called an alternating series, because the terms alternate between positive and negative values.

Example :-

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n} = -1+\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}+\cdots$$

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{2n+1} = \frac{1}{3}-\frac{2}{5}+\frac{3}{7}-\frac{4}{9}+\cdots$$

We have Leibnitz's test to check the convergence for such alternating series:

Leibnitz's test (Alternating Series test) :-

If the alternating series

$$\sum_{n=1}^{\infty}(-1)^{n+1}x_n = x_1-x_2+x_3-x_4+\cdots, \quad x_n>0$$

satisfies

(i) $x_{n+1} \le x_n$ for all n.

(ii) $\lim_{n\to\infty} x_n = 0$

Then the series $\sum_{n=1}^{\infty}(-1)^{n+1}x_n$ Converges.

Example 1: Test the convergence for the following series

$$\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}.$$

Here $x_n = \frac{1}{n}$,

$$\boxed{\lim_{n \to \infty} x_n = \lim_{n \to \infty} \frac{1}{n} = 0}$$

&

$$x_{n+1} = \frac{1}{n+1} < \frac{1}{n} = x_n$$

$$\boxed{x_{n+1} < x_n \quad \forall n}$$

Hence by Leibnitz's test the series $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$ is convergent.

Example 2: Test the convergence for the following series

$$\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2+1}.$$

Here $x_n = \frac{n}{n^2+1}$

$$\lim_{n \to \infty} x_n = \lim_{n \to \infty} \frac{n}{n^2+1} = \lim_{n \to \infty} \frac{1/n}{1+1/n^2} = 0.$$

$$\boxed{\lim_{n \to \infty} x_n = 0}$$

to check if the term x_n decrease as n increases, we use a derivative. Let

$f(x) = \frac{x}{x^2+1}$, we have $f(n) = x_n$

$$f'(x) = \frac{(x^2+1) - x(2x)}{(x^2+1)^2}$$

$$f'(x) = \frac{1-x^2}{(x^2+1)^2} < 0 \quad \forall\ x > 1.$$

Hence this function is decreasing as 'x' increases, for $x > 1$, we must have

$$x_{n+1} < x_n \text{ for } n \geq 1.$$

Therefore, by Leibnitz's test, the alternating series $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2+1}$ Converges.

Exercises:- Test the following series for convergence

(a) $\sum_{n=1}^{\infty} (-1)^n \frac{2n^2}{n^2+1}$ (Try yourself)

(b) $\sum_{n=1}^{\infty} (-1)^n \frac{1}{\ln}$ (Try yourself)

(c) $\sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{n^2}$ (")

(d) $\sum_{n=1}^{\infty} (-1)^n \cos\left(\frac{\pi}{n}\right)$ (")

(e) $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^2}$ (Try yourself)

(f) $\sum_{n=2}^{\infty} (-1)^n \left(\frac{1}{\log n}\right)$ Converge. (Try yourself.)

Absolute and Conditional convergence

A series $\sum u_n$ is called absolute convergence if series $\sum |u_n|$ converges.

Example: Test the absolute convergence of the series $\sum \frac{(-1)^{n+1}}{n^2}$.

$u_n = \frac{(-1)^{n+1}}{n^2}$,

$\sum |u_n| = \sum \frac{1}{n^2}$, series is convergent

hence $\sum \frac{(-1)^{n+1}}{n^2}$ is absolute convergent.

Exercise:- Test the absolute convergence of following series:

(a) $\sum \frac{(-1)^{n+1}}{n}$, (b) $\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$

(c) $\sum_{n=1}^{\infty} \frac{(-4)^n}{n^2}$ (d) $\sum_{n=0}^{\infty} \left(-\frac{4}{5}\right)^n$

Conditional Convergence :

A series $\sum u_n$ is called Conditional Convergence if $\sum u_n$ converges but $\sum |u_n|$ does not converge.

Example: Classify as absolutely convergent, conditionally convergent or divergent of the series:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n+1}+\sqrt{n}}$$

Check for absolute convergence first:

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}}$$, use Limit Comparison test

Choose $y_n = \frac{1}{\sqrt{n}} = \frac{1}{n^{1/2}}$

$\sum y_n$, p-series, $p = \frac{1}{2} < 1$

$\Rightarrow \sum y_n$ diverges.

$$\lim_{n\to\infty} \frac{x_n}{y_n} = \lim_{n\to\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}} \times \frac{\sqrt{n}}{1}$$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n}}+1} = \frac{1}{2} \neq 0 < \infty$$

$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}}$ diverges, then the series $\sum \frac{(-1)^{n+1}}{\sqrt{n+1}+\sqrt{n}}$.

Still need to check for Conditional convergence

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n+1}+\sqrt{n}} , \qquad x_n = \frac{1}{\sqrt{n+1}+\sqrt{n}}$$

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}} = 0 .$$

and

$$x_{n+1} = \frac{1}{\sqrt{n+2}+\sqrt{n+1}} < \frac{1}{\sqrt{n+1}+\sqrt{n}} = x_n \quad \forall n$$

$$x_{n+1} < x_n \quad \forall n$$

hence by Leibnitz's test

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n+1}+\sqrt{n}}$$ is Convergent.

Therefore, the given series converges conditionally, but not absolutely.